Bivariate Distributions (Discrete)
Bivariate Distributions

**Univariate:** One measurement for observed items. (outcome associated with one variable). E.g.
- Waiting time \( \text{Exp} \)
- Number of successes in \( n \) trials \( \text{Bin} \)
- Number of occurrences in a unit time, etc. \( \text{Pois} \)

**Bivariate:** Use 2 variables to predict an outcome.
E.g. Predict college GPA, \( z \), using HS class rank, \( x \), and ACT score, \( y \),
\[ z = f(x, y) \]
Definition 4.1-1
Let $X$ and $Y$ be two random variables defined on a discrete space. Let $S$ denote the corresponding two-dimensional space of $X$ and $Y$, the two random variables of the discrete type. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the joint probability mass function (joint pmf) of $X$ and $Y$ and has the following properties:

(a) $0 \leq f(x, y) \leq 1$.
(b) $\sum_{(x, y) \in S} f(x, y) = 1$.
(c) $P[(X, Y) \in A] = \sum_{(x, y) \in A} f(x, y)$, where $A$ is a subset of the space $S$. 

 Roll 1 die: $X = \{1, 2, 3, 4, 5, 6\}$

Roll 2 die: $S = \{(1,1), (1,2), \ldots\}$
Discrete Bivariate Example

\[ f(x, y) = \frac{xy^2}{30}, \quad x = 1, 2, 3 \quad y = 1, 2. \]

\[ P[x=2, y=1] = \frac{2(1)^2}{30} = \frac{1}{15}. \]

(a) \( 0 \leq f(x,y) \leq 1. \)

(b) \( \sum \sum f(x,y) = 1. \)

(c) \( P[(X, Y) \in A] = \sum \sum f(x,y), \) where \( A \) is a subset of the space \( S. \)
Discrete Bivariate Example

Let $X$ and $Y$ be two discrete random variables such that their joint distribution is given below:

\[ P[X=3, Y=0] = 0.31 \]

\[ P[Y=0] = 0.73 \]

<table>
<thead>
<tr>
<th>$Y$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.31</td>
<td>0.21</td>
<td>0.21</td>
<td>0.73</td>
</tr>
<tr>
<td>1</td>
<td>0.03</td>
<td>0.04</td>
<td>0.05</td>
<td>0.12</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>0.03</td>
<td>0.04</td>
<td>0.09</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.06</td>
</tr>
</tbody>
</table>

\[ f(3,0) = 0.31 \]
Definition 4.1-2

Let \(X\) and \(Y\) have the joint probability mass function \(f(x, y)\) with space \(S\). The probability mass function of \(X\) alone, which is called the **marginal probability mass function of \(X\)**, is defined by

\[
f_X(x) = \sum_y f(x, y) = P(X = x), \quad x \in S_X,
\]

where the summation is taken over all possible \(y\) values for each given \(x\) in the \(x\) space \(S_X\). That is, the summation is over all \((x, y)\) in \(S\) with a given \(x\) value. Similarly, the **marginal probability mass function of \(Y\)** is defined by

\[
f_Y(y) = \sum_x f(x, y) = P(Y = y), \quad y \in S_Y,
\]
Marginal probability

\[ f(y) = \begin{cases} 
0.73, & y = 0 \\
0.12, & y = 1 \\
0.09, & y = 2 \\
0.06, & y = 3 
\end{cases} \]

\[ f(x) = \begin{cases} 
X & 3 & 4 & 5 \\
Y & 0.31 & 0.21 & 0.21 & 0.73 \\
1 & 0.03 & 0.04 & 0.05 & 0.12 \\
2 & 0.02 & 0.03 & 0.04 & 0.09 \\
3 & 0.01 & 0.02 & 0.03 & 0.06 \\
3 & 0.37 & 0.30 & 0.33 & 1.0 
\end{cases} \]
Independence of X and Y

X and Y are independent iff:

- for every $x \in S_x$ and $y \in S_y$,

$$P[X = x, Y = y] = P[X = x]P[Y = y]$$

i.e.,

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$
Examples

Bivariate Discrete
Let \( f(x, y) = \frac{xy^2}{30}, \) \( x = 1, 2, 3 \) \( y = 1, 2. \)

A) Find the marginal pmf of \( X: \)
\[
f_X(x) = \frac{x}{6}, \quad x = 1, 2, 3.
\]

B) Find the marginal pmf of \( Y: \)
\[
f_Y(y) = \frac{y^2}{5}, \quad y = 1, 2.
\]

C) Find \( P[X=Y] \): \[
P[(1,1), (2,2)] = \frac{9}{30}
\]

D) Are \( X \) and \( Y \) independent? (Yes)
\[
\frac{1(1)^2}{30} + \frac{2(2)^2}{30} = \frac{1}{30} + \frac{8}{30} = \frac{9}{30}
\]
Figure 4.1-3  Joint pmf \( f(x, y) = \frac{xy^2}{30} \), \( x = 1, 2, 3 \) and \( y = 1, 2 \)
Let \( f(x, y) = c(x + 2y) \), \( x = 1, 2 \) \( y = 1, 2, 3 \)

What value must the constant \( c \) take, so that \( f(x, y) \) is a valid joint pmf?

\[
\begin{align*}
\sum f(1, 1) + f(1, 2) + \ldots &= 1 \\
\sum f(1, 1) &= c(3) \\
\sum f(2, 1) &= c(2 + 2(1)) = 4c \\
\sum f(1, 2) &= c(5) \\
\sum f(1, 3) &= c(7) \\
33c &= 1 \\
\Rightarrow c &= 1/33
\end{align*}
\]
Let $f(x, y) = 6 \left( \frac{1}{4} \right)^x \left( \frac{1}{3} \right)^y$, $x = 1, 2, 3, ...$ $y = 1, 2, 3, ...$

A) Find an expression for the marginal pmf of $x$.

$$f_X(x) = 3 \left( \frac{1}{4} \right)^x$$

B) Show that the marginal pmf of $x$ is a valid probability distribution.

$$\sum_{x=1}^{\infty} 3 \left( \frac{1}{4} \right)^x = 3 \sum_{x=1}^{\infty} \left( \frac{1}{4} \right)^x = 3 \frac{\frac{1}{4}}{1 - \frac{1}{4}} = 3 \left( \frac{1}{3} \right) = 1$$